Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE REAL AUTOMORPHIC LINEAR TRANSFORMATION OF A REAL BILINEAR FORM.

BY HENRY TABER.

Presented October 14, 1903. Received October 22, 1903.

§ 1.

LET

$$\mathfrak{F} \equiv (A \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n)$$

denote a bilinear form $\Sigma_r{}_1^n \; \Sigma_s{}_1^n \; a_{rs} \, x_r \, y_s$ of non-zero determinant. Let the x's and y's be transformed respectively by the linear homogeneous transformations T and T_1, so that

$$(x_1', x_2', \ldots x_n') = (T \between x_1, x_2, \ldots x_n),$$

$$(y_1', y_2', \ldots y_n') = (T_1 \between x_1, x_2, \ldots x_n).$$

It will be assumed as the result of these substitutions that $\mathfrak{F}$ is transformed automorphically, so that

$$(A \between x_1', x_2', \ldots x_n') = (A \between x_1, x_2, \ldots x_n),$$

for which the necessary and sufficient condition is

$$\breve{T}_1 \, A \; T = A.^*$$

I shall denote the family of transformations T of the x's by

1st, Γ', when the x's and y's are contragredient, in which case we have $T_1 = \breve{T}^{-1}$;

2d, by Γ'', when the respective transformations of the x's and of the y's are conjugate, so that $T_1 = \breve{T}$;

3d, by Γ''', when the product of the respective transformations of the x's and of the y's is the identical transformation, so that $T_1 = T^{-1}$. The conditions necessary and sufficient that T shall be a transformation of Γ', Γ'', or Γ''', respectively, are then

---

* Cayley: Phil. Trans., 1858. Throughout this paper $\breve{T}$ will denote the transverse or conjugate of the linear transformation T.

$$(1)\ T^{-1} A\, T = A,$$

$$(2)\ T A\, T = A,$$

$$(3)\ \breve{T}^{-1} A\, T = A.$$

The transformations of Γ' constitute a group, as do also the transformations of Γ'''; the transformations of Γ'' that are commutative form a group.

In a paper entitled "Note on the Automorphic Linear Transformation of a Bilinear Form,"[*] I have shown that each transformation T of the family Γ' belongs to a group with a single parameter of transformations of Γ', and can thus be generated by an infinitesimal transformation of Γ'; and that a similar theorem holds for the family Γ'''. But on the other hand, for certain forms $\mathfrak{F}$, that Γ'' contains transformations (of determinant $+1$ as well as of determinant -1) that cannot be generated by an infinitesimal transformation of Γ''.[†] In this paper I consider only real forms $\mathfrak{F}$, and only real transformations of the variables x and y. Thus in what follows the matrix A and the families Γ', Γ'', Γ''' of transformations T will be assumed to be real. And with this restriction to real forms and families of real transformations I show severally that each of the families Γ', Γ'', Γ''', for certain forms $\mathfrak{F}$, contains transformations (of positive as well as of negative determinant) that cannot be generated by infinitesimal transformations of that family.[‡] A transformation of either of the families Γ which is the second power of a transformation of that family I term a transformation of the *first kind*, otherwise a transformation of the *second kind*; and, for each of the families Γ, I show that every transformation of the first kind, but no transformation of the second kind, can be generated by an infinitesimal transformation of that family. I also show that a transformation of either of the families Γ is a transformation of the first kind when no negative number is a root of the characteristic equation of the transformation.* Finally, I show that each transformation of the second kind of either of the families Γ is the $(2m + 1)^{th}$ power of a transformation of that family for any odd exponent $2m + 1$.

---

[*] These Proceedings, **31**, 181.

[†] If T is a transformation of Γ'', then $|T|^2 = 1$. That Γ'' shall contain transformations of the type mentioned and of determinant $+1$, it is sufficient that two of the real roots of the characteristic equation of $\mathfrak{F}$ shall be of opposite sign and equal in absolute value.

[‡] In order that Γ', Γ'', Γ''' shall severally contain transformations of the type mentioned, it is sufficient for Γ' that the real roots of the characteristic equation of A shall not all be distinct; for Γ'', that the ratio of two of the real roots of this equation shall be equal to -1; and for Γ''', that the ratio of two of the real roots of the equation shall be negative.

The theorems given above depend upon considerations relating to the exponential function

$$e^U \equiv 1 + U + \frac{U^2}{2!} + \frac{U^3}{3!} + \cdots$$

of the matrix or linear substitution U. This series is convergent for any finite matrix. We have

$$(e^U)^{-1} = e^{-U},$$

$$\overset{\smile}{(e^{U})} = e^{\check{U}};$$

and, if U_1 and U_2 are commutative,

$$e^{U_1} e^{U_2} = e^{U_1 + U_2},$$

in particular, for any integer m,

$$(e^U)^m = e^{mU}.$$

For any linear substitution T of non-zero determinant a polynomial $U = \Sigma^n_{0\,p}\, c_p\, T^p$ can be found such that $T = e^U$. Let $\zeta_1, \zeta_2, \ldots \zeta_\nu$, respectively of multiplicity $\mu_1, \mu_2, \ldots \mu_\nu$, be the distinct roots of the characteristic equation of T; let, moreover,

$$G_i^{(k)}(T) \equiv \left(\frac{(T - \zeta_i)^{\mu_i} - (\zeta_k - \zeta_i)^{\mu_i}}{-(\zeta_k - \zeta_i)^{\mu_i}} \right)^{\mu_k},$$

$$(i, k = 1, 2, \ldots \nu \qquad k \ddagger i)$$

$$F_i(T) \equiv G_i^{(1)}(T) \ldots G_i^{(i-1)}(T)\, G_i^{(i+1)}(T) \ldots G_i^{(\nu)}(T);$$

$$(i = 1, 2, \ldots \nu)$$

and let

$$f(T) \equiv \Sigma^\nu_{1\,i} \Bigg[\log \zeta_i + 2 m_i \sqrt{-1} + \frac{1}{\zeta_i}(T - \zeta_i) - \frac{1}{\zeta_i^2} \frac{(T - \zeta_i)^2}{2} + \frac{1}{\zeta_i^3} \frac{(T - \zeta_i)^3}{3} - \ldots + (-1)^{\mu_i - 1} \frac{1}{\zeta_i^{\mu_i - 1}} \frac{(T - \zeta_i)^{\mu_i - 1}}{\mu_i - 1} \Bigg] F_i(T),$$

where m_1, m_2, etc., are integers. Then

$$T = e^{f(T)}.$$

---

* The determinant of a transformation of the first kind is positive.

Moreover, if $\phi\,(T)$ is any polynomial in T, we have

$$\phi\,(T^{-1}) = \Sigma_{1}^{\nu}{}_{i} \Big[\phi\,(\zeta_i^{-1}) + (T_i - \zeta_i) \frac{\partial \phi\,(\zeta_i^{-1})}{\partial \zeta_i} + \frac{(T - \zeta_i)^2}{2\,!} \frac{\partial^2 \phi\,(\zeta_i^{-1})}{\partial \zeta_i^2} + \dots + \frac{(T - \zeta_i)^{\mu_i - 1}}{(\mu_i - 1)\,!} \frac{\partial^{\mu_i - 1} \phi\,(\zeta_i^{-1})}{\partial \zeta_i^{\mu_i - 1}} \Big] F_i\,(T)\,.$$

In particular, if $\zeta_k = \pm\, 1$,

$$F_k\,(T^{-1}) = F_k\,(T)\,;$$

and if $\zeta_k \ddagger \pm\, 1$, and $\zeta_{k'} = \zeta_k^{-1}$ has the same multiplicity as ζ_k, then

$$F_{k'}\,(T^{-1}) = F_k\,(T)\,.$$

We have now the following theorems:

I. If U is real and $T = e^{U\sqrt{-1}}$ is also real, then $T^2 = 1$.*

II. If T is real and no negative number is a root of the characteristic equation of T, there is a real polynomial $f\,(T)$ satisfying the condition $T = e^{f(T)}$ †

III. If T is real and each negative root other than -1 of the characteristic equation of T is paired with its inverse, there is a polynomial $f\,(T)$ satisfying the equation $T = e^{f(T)}$ and such that $f(T) - f(T^{-1})$ is real.†

---

* For if $T = e^{U\sqrt{-1}}$ is real, then

$$2\sqrt{-1}\,\Big(U - \frac{U^3}{3!} + \frac{U^5}{5!} - \dots\Big) = e^{U\sqrt{-1}} - e^{-U\sqrt{-1}} = T - T^{-1}$$

is real. Therefore, if U is real, $T - T^{-1} = 0$; that is $T^2 = 1$.

† If T is real, each imaginary root of the characteristic equation of T is paired with its conjugate imaginary. Therefore, if this equation has no negative root, $f(T)$ may be taken real by a proper choice of m_1, m_2, etc.

If the distinct negative roots of the characteristic equation are

$$\zeta_i,\ \zeta_{\nu'+i} = \zeta_i^{-1}\ (i = 1, 2, \dots \nu')$$

and $\zeta_{2\nu'+1} = -1$, and moreover (for $i = 1, 2, \dots \nu'$) ζ_i and ζ_i^{-1} have the same multiplicity, then, for a proper choice of m_1, m_2, etc., the imaginary part of the polynomial $f(T)$ is

$$\Pi\sqrt{-1}\,\Sigma_{1}^{\nu'}{}_{i}\,(F_i\,(T) + F_{\nu'+i}\,(T)) + \Pi\sqrt{-1}\,F_{2\nu'+1}\,(T)\,;$$

and, since now

$$F_i\,(T^{-1}) = F_{\nu'+i}\,(T), \quad F_{\nu'+i}\,(T^{-1}) = F_i\,(T),$$
$$(i = 1, 2, \dots \nu')$$
$$F_{2\nu'+1}\,(T^{-1}) = F_{2\nu'+1}\,(T),$$

therefore $f(T) - f(T^{-1})$ will be real.

IV. If T is real and each negative root of the characteristic equation of T is in absolute value less than unity, there is a real polynomial $\mathrm{f}(T)$ satisfying the equation $1 + T = e^{\mathrm{f}(T)}$.

Finally, from the identity,

$$e^{-AUA^{-1}} A e^{U} = A e^{-U} A^{-1} \cdot A e^{U} = A e^{-U} e^{U} = A,$$

we derive the theorems:

V. If U satisfies the equation

$$U = AUA^{-1} \tag{4}$$

and $T = e^{U}$ is real, in particular if U is real, then T is a transformation of Γ'.*

VI. If U satisfies the equation

$$U = -A\,UA^{-1} \tag{5}$$

and $T = e^{U}$ is real, in particular if U is real, then T is a transformation of Γ''.

VII. If U satisfies the equation

$$\check{U} = AUA^{-1}, \tag{6}$$

and $T = e^{U}$ is real, in particular if U is real, then T is a transformation of Γ'''.

§ 2. $T^{-1} A\, T = A.$

If U is real and satisfies the equation

$$U = A\,UA^{-1}, \tag{4}$$

then, by Theorem V, every transformation of the group $e^{\zeta U}$, with real parameter ζ, is a transformation of Γ'; in particular the infinitesimal transformation $e^{\delta\zeta U}$ of this group is a transformation of Γ'.

If the real infinitesimal transformation $e^{\delta\zeta U} = 1 + \delta\zeta U$ is a transformation of Γ', then first U is real; moreover,

$$A + \delta\zeta(-UA + AU) = (1 - \delta\zeta U)\,A\,(1 + \delta\zeta U) = e^{-\delta\zeta U} A\, e^{\delta\zeta U} = A\,,$$

and therefore $UA = AU$, that is, U satisfies equation (4). Therefore, for any real scalar ζ, the transformation $e^{\zeta U}$ generated by $e^{\delta\zeta U}$ is a transformation of Γ'. Whence it follows that *every transformation generated*

---

* In this case the application of the above identity is unnecessary; for $UA = AU$, and therefore $T^{-1} A\, T = e^{-U} A\, e^{U} = A\, e^{-U} e^{U} = A$.

by an infinitesimal transformation of Γ' *is the* m^{th} *power for any exponent m of a transformation of* Γ' *; in particular, every such transformation is the second power of a transformation of* Γ', *and is therefore of the first kind.**

Conversely, *every transformation of the first kind can be generated by an infinitesimal transformation of* Γ'. For let T be any transformation of Γ', then

$$T = A\,T A^{-1}. \tag{7}$$

Let V and $W\sqrt{-1}$, respectively, be the real and imaginary parts of the polynomial $U = f(T)$, satisfying the equation $T = e^U$. Since T is real, both V and W are polynomials in T, and therefore commutative. Let

$$T_1 = e^V, \quad T_0 = e^{W\sqrt{-1}}.$$

Then since V, and therefore $T_1 = e^V$, is real, and since

$$T = e^U = e^{V + W\sqrt{-1}} = e^V e^{W\sqrt{-1}} = T_1\,T_0$$

is also real, it follows that $T_0 = T_1^{-1}\,T$ is real; and therefore, by Theorem I,

$$T_0^{\,2} = (e^{W\sqrt{-1}})^{\,2} = 1.$$

Whence,

$$T^2 = (T_1\,T_0)^{\,2} = T_1^{\,2}\,T_0^{\,2} = T_1^{\,2}.$$

But, since $V = \phi\,(T)$ is a polynomial in T, we have by equation (7)

$$V = \phi\,(T) = \phi\,(A\,T A^{-1}) = A\,\phi\,(T)\,A^{-1} = A\,V A^{-1};$$

and therefore, since V is real, $e^{\zeta V}$ for any real scalar ζ, is generated by the infinitesimal transformation $e^{\delta\zeta V}$ of Γ'. In particular,

$$T^2 = T_1^{\,2} = (e^V)^{\,2} = e^{2V}$$

is generated by the infinitesimal transformation $e^{\delta\zeta V}$ of Γ'.

A transformation T of Γ' *is a transformation of the first kind if the characteristic equation of T has no negative root.* For then, by Theorem II, there is a real polynomial $U = f(T)$ satisfying the equation $T = e^U$. In this case $V = U$, $W = 0$; and U satisfies equation (4). Therefore, T is the second power of the transformation $e^{\frac{1}{2}U}$ of Γ', and is generated by the infinitesimal transformation $e^{\delta\zeta U}$ of Γ'.

---

* See p. 308.

The characteristic equation of a transformation of Γ' of the first kind may have negative roots. But in such case the numbers belonging to each of the negative roots of this equation are all even; and therefore, for each negative root of this equation, the elementary divisors (*elementar Theiler*) of the characteristic function corresponding to such root occur in pairs with equal exponent.*

No transformation of Γ' with negative determinant is of the first kind. Let $n = 2$, and

$$\mathfrak{J} \equiv \begin{pmatrix} 1, & b \\ 0, & 1 \end{pmatrix} \between x_1, x_2 \between y_1, y_2) \equiv (x_1 y_1 - x_2 y_2) + b\, x_2 y_1 .$$

The form $\mathfrak{J}$ is transformed automorphically if

$$(x_1', x_2') = T(x_1, x_2) = \begin{pmatrix} -1, & 1 \\ 0, & -1 \end{pmatrix} \between x_1\ x_2),$$

$$(y_1', y_2') = \breve{T}^{-1}(y_1, y_2) = \begin{pmatrix} -1, & 0 \\ -1, & -1 \end{pmatrix} \between y_1, y_2).$$

We have $|T| = +1$, but T is a transformation of the second kind. Whence it follows, for any value of n, that there are forms $\mathfrak{J}$ such that Γ' contains transformations of the second kind with positive determinant.†

By definition no transformation of the second kind is an even power of a transformation of Γ'; but *every transformation of* Γ' *of the second kind is the* $(2m + 1)^{th}$ *power, for any odd exponent* $2m + 1$, *of a transformation of* Γ'. Thus, let T be any transformation of Γ', and, as before, let V and W be real polynomials in T satisfying the equation

$$T = e^{V + W\sqrt{-1}}.$$

As shown above, for any real scalar ζ, $e^{\zeta V}$ is a transformation of Γ'; and, therefore, so also is $e^{W\sqrt{-1}} = Te^{-V}$. Consequently

---

* For the roots of the characteristic equation of T^2 are the squares of the roots of the characteristic equation of T; and, if T is real, each imaginary root of the characteristic equation of T is paired with its conjugate imaginary. Compare These Proceedings, **31**, 189.

For the relation between the numbers belonging to the roots of the characteristic equation of a linear transformation T and the exponents of the elementary divisors of the characteristic function of T, see Bull. Am. Math. Soc., 2d series, **3**, 156.

† The condition, given on p. 308, sufficient that Γ' shall contain transformations of the second kind, is readily obtained.

$$e^{\zeta V + W\sqrt{-1}} = e^{\zeta V} e^{W\sqrt{-1}}$$

is a transformation of Γ′.* Moreover,

$$e^{2W\sqrt{-1}} = (e^{W\sqrt{-1}})^2 = 1\,;$$

and therefore, for any integer m,

$$(e^{W\sqrt{-1}})^{2m+1} = (e^{2W\sqrt{-1}})^m e^{W\sqrt{-1}} = e^{W\sqrt{-1}}.$$

Whence we have

$$T = e^V e^{W\sqrt{-1}} = (e^{\frac{1}{2m+1}V})^{2m+1} (e^{W\sqrt{-1}})^{2m+1}$$
$$= (e^{\frac{1}{2m+1}V} e^{W\sqrt{-1}})^{2m+1} = (e^{\frac{1}{2m+1}V + W\sqrt{-1}})^{2m+1}.$$

§3. $TAT = A$.

If U is real and satisfies the equation

$$(5) \qquad U = -A\,UA^{-1},$$

then, by Theorem VI, every transformation of the group $e^{\zeta U}$, with real parameter ζ, is a transformation of Γ″: in particular, the infinitesimal transformation $e^{\delta\zeta U}$ of the group is a transformation of Γ″.

If the real infinitesimal transformation $e^{\delta\zeta U} = 1 + \delta\,\zeta\,U$ is a transformation of Γ″, then first U is real; moreover

$$A + \delta\zeta(UA + AU) = (1 + \delta\zeta\,U)\,A\,(1 + \delta\zeta\,U) = e^{\delta\zeta U} A\, e^{\delta\zeta U} = A\,;$$

and therefore $UA + AU = U$, that is, U satisfies equation (5). Therefore, for any real scalar ζ, the transformation $e^{\zeta U}$ generated by $e^{\delta\zeta U}$ is a transformation of Γ″. Whence it follows that *every transformation generated by an infinitesimal transformation of* Γ″ *is the mth power for any exponent m, of a transformation of this family; in particular, every such transformation is the second power of a transformation of* Γ″, *and is therefore of the first kind.*

Conversely, *every transformation of the first kind can be generated by an infinitesimal transformation of* Γ″. For let T be any transformation of Γ″, then

---

* Since the transformations of Γ′ form a group, if T and T_1 are transformations of Γ′, so also is $T\,T_1$.

$$\begin{aligned} T &= A\,T^{-1}A^{-1}, \\ T^{-1} &= A\,TA^{-1}; \end{aligned} \tag{8}$$

and therefore each root, other than ± 1, of the characteristic equation of T is paired with its inverse, so that if $\zeta_i \mp \pm 1$ is a root of this equation, $\zeta_{i'} = \zeta_i^{-1}$ is also a root, and the numbers belonging to ζ_i and $\zeta_{i'}$ (in particular the multiplicity of these roots) are the same. Therefore, by Theorem III, there is a polynomial $f(T)$ satisfying the equation $T = e^{f(T)}$ and such that $f(T) - f(T^{-1})$ is real. Let now

$$2\,U_0 = f(T) + f(T^{-1}), \qquad 2\,U_1 = f(T) - f(T^{-1});$$

and let

$$T_0 = e^{U_0}, \qquad T_1 = e^{U_1}.$$

We have by (8)

$$\begin{aligned} T_0^{\,2} &= e^{f(T)+f(T^{-1})} \\ &= e^{f(T)}\,e^{f(T^{-1})} \\ &= e^{f(T)}\,e^{f(ATA^{-1})} \\ &= e^{f(T)}\,e^{Af(T)A^{-1}} \\ &= e^{f(T)} \cdot A\,e^{f(T)}\,A^{-1} \\ &= T\,A\,T \cdot A^{-1} = A\,A^{-1} = 1. \end{aligned}$$

Therefore,

$$T^2 = (T_0\,T_1)^2 = T_0^{\,2}\,T_1^{\,2} = T_1^{\,2}.$$

But, by equation (8),

$$\begin{aligned} U_1 &= \tfrac{1}{2}\,(f(T) - f(T^{-1})) \\ &= \tfrac{1}{2}\,(f(A\,T^{-1}A^{-1}) - f(A\,TA^{-1})) \\ &= \tfrac{1}{2}\,A(f(T^{-1}) - f(T))\,A^{-1} = -\,A\,U_1\,A^{-1}; \end{aligned}$$

and therefore, since U_1 is real, it follows that $e^{\zeta U_1}$, for any real scalar ζ, is generated by the infinitesimal transformation $e^{\delta\zeta U_1}$ of Γ''. In particular

$$T^2 = T_1^{\,2} = (e^{U_1})^2 = e^{2U_1}$$

is generated by $e^{\delta\zeta U_1}$.

A transformation T *of* Γ'' *is of the first kind if the characteristic* *equation of T has no negative root.* For in this case, since $|1 + T| \neq 0$, we may put

$$S = \frac{1 - T}{1 + T};$$

whence we derive

$$T = \frac{1 - S}{1 + S},$$

and by (8)

$$(9) \qquad A S A^{-1} = \frac{1 - A T A^{-1}}{1 + A T A^{-1}} = \frac{1 - T^{-1}}{1 + T^{-1}} = \frac{T - 1}{T + 1} = - S.$$

If ζ_1, ζ_2, etc., are the roots of the characteristic equation of T, the roots of the characteristic equation of S are $\dfrac{1 - \zeta_i}{1 + \zeta_i}$ for $i = 1, 2$, etc.; and since no negative number is a root of the characteristic equation of T, the real roots of the characteristic equation of S are in absolute value less than unity. Therefore, by Theorem IV, there is a real polynomial $\mathrm{f}(S)$ satisfying the equation

$$1 + S = e^{\mathrm{f}(S)}.$$

From equation (9) we derive

$$1 - S = 1 + A S A^{-1} = A (1 + S) A^{-1} = e^{A\mathrm{f}(S)A^{-1}} = e^{\mathrm{f}(ASA^{-1})} = e^{\mathrm{f}(-S)};$$

and therefore, if $U = \mathrm{f}(-S) - \mathrm{f}(S)$,

$$T = \frac{1 - S}{1 + S} = e^{\mathrm{f}(-S)} e^{-\mathrm{f}(S)} = e^{\mathrm{f}(-S) - \mathrm{f}(S)} = e^{U}.$$

But, by equation (9),

$$\begin{aligned} U = \mathrm{f}(-S) - \mathrm{f}(S) &= \mathrm{f}(A S A^{-1}) - \mathrm{f}(-A S A^{-1}) \\ &= A (\mathrm{f}(S) - \mathrm{f}(-S)) A^{-1} = - A U A^{-1}; \end{aligned}$$

and therefore, since U is real, $e^{\zeta U}$, for any real scalar ζ, is a transformation of Γ''. Consequently, T is the second power of a transformation $e^{\frac{1}{2}U}$ of Γ'', and is generated by the infinitesimal transformation $e^{\delta\zeta U}$ of Γ''.

The characteristic equation of a transformation of the first kind may have negative roots. But in such case, the numbers belonging to each of these roots are all even.*

---

* Cf. note, p. 313.

Every transformation of Γ'' with negative determinant is of the second kind. Let $n = 2$, and let

$$\mathfrak{F} \equiv \left(\begin{array}{cc} 1, & 0 \\ 0, & -1 \end{array}\right. \between x_1, x_2 \between y_1, y_2) \equiv x_1 y_1 - x_2 y_2.$$

The form $\mathfrak{F}$ is transformed automorphically if

$$(x_1', x_2') = T(x_1, x_2) = \left(\begin{array}{cc} -1, & 1 \\ 0, & -1 \end{array}\right. \between x_1, x_2)$$

$$(y_1', y_2') = \breve{T}(y_1, y_2) = \left(\begin{array}{cc} -1, & 0 \\ 1, & -1 \end{array}\right. \between y_1, y_2).$$

We have $|T| = +1$, but T is a transformation of the second kind. Whence, for any value of n, it follows that there are forms $\mathfrak{F}$ such that Γ'' contains transformations of the second kind with determinant $+1$.* Again, the form

$$\mathfrak{F} \equiv \left(\begin{array}{cccc} 0, & 0, & b, & d \\ 0, & 0, & 0, & -b \\ a, & c, & 0, & 0 \\ 0 & -a, & 0, & 0 \end{array}\right. \between x_1, x\ , x_3, x_4 \between y_1, y_2, y_3, y_4) \equiv a(x_1 y_3 - x_2 y_4) + b(x_3 y_1 - x_4 y_2) + c x_2 y_3 + d x_4 y_1$$

is transformed automorphically by the transformation

$$T = \left(\begin{array}{cccc} -\lambda^2, & 1, & 0, & 0 \\ 0, & -\lambda^2, & 0, & 0 \\ 0, & 0, & -\lambda^{-2}, & \lambda^{-4} \\ 0, & 0, & 0, & -\lambda^{-2} \end{array}\right)$$

of Γ''; and T is a transformation of the second kind if $\lambda \neq \pm 1$.

By definition no transformation of the second kind is an even power of any transformation of Γ''; but *every transformation of the second kind is the* $(2m + 1)^{th}$ *power, for any odd exponent* $2m + 1$, *of a transformation of* Γ''. Thus let T be any transformation of Γ''; and, as before, let $f(T)$ be a polynomial satisfying the equation $T = e^{f(T)}$, and such that $f(T) - f(T^{-1})$ is real. Then, as shown above, if

$$2\,U_0 = f(T) + f(T^{-1}), \quad 2\,U_1 = f(T) - f(T^{-1}),$$

---

* The condition, given in note on p. 308, as sufficient that Γ'' shall contain transformations, is readily proved.

$e^{\zeta U_1}$, for any real scalar ζ, is a transformation of Γ''; and therefore

$$e^{U_0} = T e^{-U_1}$$

is a transformation of Γ''. Whence it follows that

$$e^{U_0 + \zeta U_1} = e^{U_0} e^{\zeta U_1},$$

for any real scalar ζ, is a transformation of Γ''.* Moreover, we have

$$e^{2U_0} = (e^{U_0})^2 = 1;$$

and therefore for any integer m

$$(e^{U_0})^{2m+1} = (e^{2U_0})^m e^{U_0} = e^{U_0}.$$

Wherefore,

$$T = e^{U_0} e^{U_1} = (e^{U_0})^{2m+1} (e^{\frac{1}{2m+1} U_1})^{2m+1} = (e^{U_0} e^{\frac{1}{2m+1} U_1})^{2m+1} = (e^{U_0 + \frac{1}{2m+1} U_1})^{2m+1}.$$

§ 4. $\breve{T}^{-1} A T = A$.

If U is real and satisfies the equation

$$\breve{U} = A U A^{-1}, \tag{5}$$

then, by Theorem VI, every transformation of the group $e^{\zeta U}$, with real parameter ζ, is a transformation of Γ'''. In particular, the infinitesimal transformation $e^{\delta \zeta U}$ of the group is a transformation of Γ'''.

If the real infinitesimal transformation $e^{\delta \zeta U} = 1 + \delta \zeta U$ is a transformation of Γ''', then first U is real; moreover,

$$A + \delta \zeta (- \breve{U} A + A U) = (1 - \delta \zeta \breve{U}) A (1 + \delta \zeta U) = e^{-\delta \zeta \breve{U}} A e^{\delta \zeta U} = A,$$

and therefore $- \breve{U} A + A U = 0$, that is, U satisfies equation (5). Therefore, for any real scalar ζ, the transformation $e^{\zeta U}$ generated by $e^{\delta \zeta U}$ is a transformation of Γ'''. Whence it follows that *every transformation generated by an infinitesimal transformation of* Γ''' *is the* m^{th} *power for any exponent* m *of a transformation of* Γ'''. *In particular, every such transformation is the second power of a transformation of* Γ''', *and is therefore of the first kind.*

---

[*] The transformations of Γ'' that are commutative form a group. Thus, if T and $T^{(1)}$ are transformations of Γ'' and $T T^{(1)} = T^{(1)} T$, then $T T^{(1)}$ is also a transformation of Γ''.

Conversely, *every transformation of the first kind can be generated by an infinitesimal transformation of* Γ'''. For let T be any transformation of Γ''', then

$$(10) \qquad \breve{T} = A\,T A^{-1}.$$

Let V and $W\sqrt{-1}$, respectively, be the real and imaginary parts of the polynomial $U = f(T)$ satisfying the equation $T = e^U$. Then, since T is real, both V and W are polynomials in T, and therefore commutative. Let

$$T_1 = e^V, \quad T_0 = e^{W\sqrt{-1}}.$$

Then since V, and therefore $T_1 = e^V$, is real, and since

$$T = e^{V + W\sqrt{-1}} = e^V e^{W\sqrt{-1}} = T_1\,T_0$$

is also real, it follows that $T_0 = T_1^{-1}\,T_0$ is real; and therefore, by Theorem I,

$$T_0^{\,2} = (e^{W\sqrt{-1}})^2 = 1.$$

Wherefore,

$$T^2 = (T_1\,T_0)^2 = T_1^{\,2}\,T_0^{\,2} = T_1^{\,2}.$$

But, since $V = \phi\,(T)$ is a polynomial in T, we have by (10)

$$\breve{V} = \phi\,(\breve{T}) = \phi\,(A\,T A^{-1}) = A\,\phi\,(T)\,A^{-1} = A\,V A^{-1};$$

and therefore, since V is real, $e^{\zeta V}$ for any real scalar ζ is generated by the infinitesimal transformation $e^{\delta\zeta V}$ of Γ'''. In particular,

$$T^2 = T_1^{\,2} = (e^V)^2 = e^{2V}$$

is generated by $e^{\delta\zeta V}$.

A transformation T *of* Γ''' *is of the first kind if the characteristic equation of* T *has no negative root.* For then, by Theorem II, there is a real polynomial $U = f\,(T)$ satisfying the equation $T = e^U$. In this case $V = U$, $W = 0$; and U satisfies equation (5). Therefore T is the second power of the transformation $e^{\frac{1}{2}U}$ of Γ''', and is generated by the infinitesimal transformation $e^{\delta\zeta U}$ of Γ'''.

The characteristic equation of a transformation of the second kind may have negative roots. But in such case the numbers belonging to the negative root of this equation are all even.*

Every transformation of Γ''' with negative determinant is a transformation of the second kind. Let $n = 2$, and let

---

* Cf. p. 312.

$$\mathfrak{F} \equiv \left(\begin{matrix}0, & 1\\ 1, & b\end{matrix} \between x_1, x_2 \between y_1, y_2\right) \equiv (x_1 y_2 + x_2 y_1) + b\, x_2 y_2.$$

The form $\mathfrak{F}$ is transformed automorphically if

$$(x_1', x_2') = (T \between x_1, x_2) = \left(\begin{matrix}-1, & 1\\ 0, & -1\end{matrix} \between x_1, x_2\right)$$

$$(y_1', y_2') = (T^{-1} \between y_1, y_2) = \left(\begin{matrix}-1, & -1\\ 0, & -1\end{matrix} \between y_1, y_2\right).$$

We have $|T| = +1$, but T is a transformation of the second kind. Whence it follows, for any value of n, that there are bilinear forms $\mathfrak{F}$ such that Γ''' contains transformations of the second kind with positive determinant.* The form

$$\mathfrak{F} \equiv \left(\begin{matrix}-a^2, & 0\\ 0, & 1\end{matrix} \between x_1, x_2 \between y_1, y_2\right) \equiv -a^2 x_1 y_1 + x_2 y_2$$

is transformed automorphically if

$$(x_1', x_2') = (T \between x_1, x_2) = \left(\begin{matrix}-1 + a\lambda, & \lambda\\ -a^2\lambda, & -1 - a\lambda\end{matrix} \between x_1, x_2\right),$$

$$(y_1', y_2') = (T^{-1} \between y_1, y_2) = \left(\begin{matrix}-1 - a\lambda, & -\lambda\\ a^2\lambda, & -1 + a\lambda\end{matrix} \between y_1, y_2\right);$$

and T, of determinant $+1$, is a transformation of the second kind if λ ‡ 0.

Every transformation Γ''' *of the second kind is the* $(2m + 1)^{th}$ *power of a transformation of* Γ''' *for any odd exponent* $2m + 1$. Thus let T be any transformation of Γ'''. If V and W are real polynomials in T satisfying the equation $T = e^{V + W\sqrt{-1}}$, we may then show, precisely as for the similar theorem in the case of the family Γ' that $e^{\frac{1}{2m+1} V + W\sqrt{-1}}$ is a transformation of Γ''', and that

$$T = e^{V + W\sqrt{-1}} = \left(e^{\frac{1}{2m+1} V + W\sqrt{-1}}\right)^{2m+1}.\dagger$$

CLARK UNIVERSITY, WORCESTER, MASSACHUSETTS.

---

* For a sufficient condition that Γ''' shall contain a transformation of the second kind see p. 308.

† See p. 313. The family of transformations Γ''' also constitute a group.